

NBI-HE 96-61
hep-th/9610176
October 1996

STRING SOLUTIONS TO SUPERGRAVITY

A. K. Tollstén¹

The Niels Bohr Institute, Blegdamsvej 17,
DK-2100 Copenhagen Ø, Denmark

Abstract

We find the complete solution to ten-dimensional supergravity coupled to a three-form field strength, given the “standard ansatz” for the fields, and show that in addition to the well-known elementary and solitonic (heterotic) string solutions, one of the possibilities is an (unstable) elementary type I string solution.

¹email tollsten@nbi.dk

1 Introduction

A necessary condition for strong-weak coupling duality between two theories is that the elementary states of one theory turn up as soliton-like states in the other. This condition is satisfied for most of the conjectured dualities in 10 and 11-dimensional theories. However, an important exception is the heterotic-type I duality in $d=10,9,8$ [1, 2, 3]. Here, the heterotic string does indeed turn up as a stable solution to the low energy effective action of type I theory [4, 5], while there is no such type I solution to the low energy effective action of the heterotic string. This is perhaps not very surprising. All other stable $p-1$ -brane solutions turning up in various dualities are *closed*, and, as we shall see for the string case, the solutions can be interpreted as a compactification (possibly with infinite compactification radii) of $p-1$ coordinates, around which the $p-1$ -brane winds. Of course, no such topological argument will hold in the open string case. On the contrary, a solution consisting of an infinitely long open string with constant energy/length unit is obviously unstable.

In this talk we review the string solutions to ten-dimensional supergravity. We first find the general solution to the equations of motion given the “standard ansatz”. These solutions make the equations of motion singular at what can be interpreted as the location of a string source. Requiring also this source to be dynamical takes us back to the well-known supersymmetric fundamental string solution [6]. In addition, there are other solutions, representing the field configuration around a fixed source. One of these shows the correct behaviour under Weyl rescaling to be a type I solution. It has no conserved charge, nor does it preserve supersymmetry, so there is no reason to believe that it is stable.

Further details of the calculation, notation and conventions are published elsewhere [7].

2 The string solution of ten-dimensional supergravity

We would like to study the bosonic part of the combined string supergravity action in ten dimensions, written in a metric explicitly rescaled with $e^{a\phi}$ with an arbitrary constant a

$$S = \frac{1}{\kappa^2} \int d^{10}x \sqrt{-g} e^{4(a+\frac{1}{3})\phi} \left[R - 9a \left(a + \frac{2}{3} \right) \partial\phi^2 + \frac{3}{2} e^{-2(a+\frac{4}{3})\phi} H^2 \right] \\ - \frac{T_2}{2} \int d^2\xi \left[\sqrt{-\gamma} \gamma^{ij} \partial_i X^M \partial_j X^N e^{(b+\frac{4}{3})\phi} g_{MN} \right. \\ \left. + 2\varepsilon^{ij} \partial_i X^M \partial_j X^N B_{MN} \right]. \quad (1)$$

Here $a = -\frac{4}{3}$ gives us the usual (heterotic) string metric, $a = -\frac{1}{3}$ the Einstein metric, and $a = \frac{2}{3}$ the type I string metric.

For $b = a$ we have a heterotic string source, while $b = a - 2$ and, no B_{MN} term in the string part of the action will give us a type I source. Since we only consider solutions with the Yang-Mills fields identically zero, we omit the $\text{tr} F^2$ term in the action just like we do with all the fermionic terms.

The variation of (1) with respect to g_{MN} , B_{MN} , ϕ , γ_{ij} , and X^M , gives us the supergravity equations of motion, with δ -function sources at the location of the string, and the equations of motion for the string degrees of freedom.

To find a string solution, we split up the coordinates ($M = 0, 1, \dots, 9$)

$$x^M = (x^\mu, y^m) \quad (2)$$

where $\mu = 0, 1$ and $m = 2, \dots, 9$, and make the ansatz

$$ds^2 = e^{2A} \eta_{\mu\nu} dx^\mu dx^\nu - e^{2B} \delta_{mn} dy^m dy^n, \quad (3)$$

$$B_{\mu\nu} = \gamma \frac{\varepsilon_{\mu\nu}}{\sqrt{g_2}} e^C, \quad (4)$$

with $g_2 = -\det g_{\mu\nu}$. All other fields are put equal to zero, and the only coordinate dependance is on $y = \sqrt{\delta_{mn} y^m y^n}$. We need a numerical constant γ in the definition of $B_{\mu\nu}$ because of our nonstandard normalization. The string coordinate $X^M(\xi)$ is split up in the same way as the coordinates (2), and we make the static gauge choice $X^\mu = \xi^\mu$, and assume $Y^m = \text{constant}$, put to zero for simplicity.

The γ_{ij} equation now immediately expresses γ_{ij} as a function of the metric and the string coordinates. To find the general solution to the remaining equations of motion it turns out to be useful to make the field redefinitions

$$X = 2A + 6B + 4 \left(a + \frac{1}{3} \right) \phi, \quad (5)$$

$$Y = 2A + \left(a - \frac{8}{3} \right) \phi, \quad (6)$$

$$Z = 2A + \left(a + \frac{4}{3} \right) \phi. \quad (7)$$

The equations of motion are then equivalent to

$$\begin{aligned} e^{X-2A} \left[\frac{7}{6} (\nabla^2 X + \frac{1}{2} X'^2) - \frac{1}{6} (\nabla^2 Y + X'Y' - \frac{1}{2} Y'^2) \right. \\ \left. - \frac{1}{2} (\nabla^2 Z + X'Z' - \frac{1}{2} Z'^2) + \gamma^2 e^{-2Z} \partial (e^{2A+C})^2 \right] \\ = -\kappa^2 T_2 e^{Z-2A+(b-a)\phi} \delta^8(y), \end{aligned} \quad (8)$$

$$e^{X-2B} \left[X'' + \frac{13X'}{y} + X'^2 \right] = 0, \quad (9)$$

$$e^{X-2B} \left[7 \left(X'' - \frac{1}{12} X'^2 \right) + \frac{13}{12} Y'^2 + \frac{13}{4} Z'^2 - 13\gamma^2 e^{-2Z} \left(\partial e^{2A+C} \right)^2 \right] = 0, \quad (10)$$

$$\gamma \frac{1}{y^7} \partial \left(y^7 e^{X-2Z} \partial e^{2A+C} \right) = \kappa^2 T_2 \delta^8(y), \quad (11)$$

$$\begin{aligned} e^X & \left[\frac{14}{3} \left(a + \frac{1}{3} \right) (\nabla^2 X + \frac{1}{2} X'^2) - \frac{1}{6} \left(a - \frac{8}{3} \right) (\nabla^2 Y + X' Y') \right. \\ & + \frac{1}{3} \left(a + \frac{1}{3} \right) Y'^2 - \frac{1}{2} \left(a + \frac{4}{3} \right) (\nabla^2 Z + X' Z') + \left(a + \frac{1}{3} \right) Z'^2 \\ & \left. - 2\gamma^2 \left(a - \frac{2}{3} \right) e^{-2Z} \left(\partial e^{2A+C} \right)^2 \right] \\ & = -\kappa^2 T_2 \left(b + \frac{4}{3} \right) e^{Z+(b-a)\phi} \delta^8(y), \end{aligned} \quad (12)$$

$$\partial e^{Z+(b-a)\phi} = 2\gamma \partial e^{2A+C}. \quad (13)$$

First we now to solve all equations for $y > 0$, and only afterwards we consider the singularity structure at $y = 0$ to find a consistent choice for the values of the integration constants. In the following, a subscript zero will always denote the value of the function in question at infinity, and $K, L \dots$ are integration constants. Choosing $\gamma = \frac{1}{2}$ (cf (13) and [7]), and assuming $L \neq 0$ (solutions with a conserved charge), the supergravity equations have the solution

$$e^X = e^{X_0} + \frac{K}{y^{12}}, \quad (14)$$

$$e^X Y' = \frac{M}{y^7}, \quad (15)$$

$$\frac{Z'}{\left(L^2 e^{2Z} - \frac{M^2}{3} - 7 \cdot 48 K e^{X_0} \right)^{1/2}} = \pm \frac{e^{-X}}{y^7}, \quad (16)$$

$$e^{2A+C} = \pm \frac{\left(L^2 e^{2Z} - \frac{M^2}{3} - 7 \cdot 48 K e^{X_0} \right)^{1/2}}{L} + \text{constant}. \quad (17)$$

Equations 15 and 16 can be integrated once more, and 16 solved for e^{-Z} in terms of y , but the exact form depends on the values of the integration constants, so we will not do this for the general case. The explicit form for $K < 0$ (for generic dimensions) can be found in a paper by Lü et al. [8]. One of the conditions for a supersymmetric solution (see below) is $X' = 0$, so we immediately see that our nontrivial candidate to a generalization cannot preserve supersymmetry.

The equation of motion for the string source we have so far neglected gives us

$$e^{2A+C} = e^{Z+(b-a)\phi} + \text{constant}. \quad (18)$$

The precise value of the constant is not very interesting since it does not affect $H_{m\mu\nu}$.

As implied by Dabholkar et al. [4, 9], this type of solution really requires a compactification. This comes about as follows: Varying the supergravity part of our action with respect to some field, we obtain the usual equations of motion, while the string part of the action gives terms of the form $\int d^2\xi \delta^{10}(X^\mu(\xi) - x^\mu) \times$ fields. With our ansatz this becomes $\int d\tau \delta(\tau - x^0) \int_0^{2\pi} d\sigma \delta(\sigma - x^1) \delta^8(y)$. The τ -integration immediately gives a factor one, while it is not at all obvious that this, which has always been assumed, should also be the case for the σ -part. To obtain a one also here, all possible values of x^1 must lie within the interval of possible values of σ , $0 \leq x^1 < 2\pi(\times R)$. The obvious interpretation is that x^1 is compactified, and that the source string winds around it once. We can then also consider strings winding n times around the compactified coordinate. Then n different values of σ correspond to each x^1 , and we obtain $\int_0^{2\pi} d\sigma \delta(\sigma - x^1) = n$. The generalization to higher branes is obvious.

3 The heterotic string solution

We will study the solution for $L \neq 0$ given in Section 2, and show that this is the one corresponding to the elementary string solution [6]. By a full analysis of the zero modes, it can be shown that this is indeed a heterotic string [4, 5].

In order to satisfy equation 18 for $L \neq 0$, we must choose

$$b = a \tag{19}$$

$$K = -\frac{M^2 e^{-X_0}}{7 \cdot 144} \tag{20}$$

This is inserted into our equations, which can then be integrated to

$$Y = \begin{cases} Y_0 + \frac{\sqrt{7}M}{|M|} \log \left| \frac{y^6 - \frac{|M|e^{-X_0}}{\sqrt{7 \cdot 12}}}{y^6 + \frac{|M|e^{-X_0}}{\sqrt{7 \cdot 12}}} \right| & M \neq 0 \\ Y_0 & M = 0 \end{cases} \tag{21}$$

and

$$e^{-Z} = \begin{cases} e^{-Z_0} - \frac{\sqrt{7}L}{|M|} \log \left| \frac{y^6 - \frac{|M|e^{-X_0}}{\sqrt{7 \cdot 12}}}{y^6 + \frac{|M|e^{-X_0}}{\sqrt{7 \cdot 12}}} \right| & M \neq 0 \\ e^{-Z_0} + \frac{Le^{-X_0}}{6y^6} & M = 0 \end{cases} . \tag{22}$$

A study of the singularities at $y = 0$ yields

$$e^X Y' = M \Omega_7 f', \tag{23}$$

$$e^X \partial e^{-Z} = -L \Omega_7 f', \tag{24}$$

where $\nabla^2 f = \delta^8(y)$, and Ω_7 is the volume of the seven-sphere. The nonvanishing parts of the equations of motion at $y = 0$ are then

$$e^{-2A} \left[-\frac{1}{6}M - \frac{1}{2}Le^Z \right] \Omega_7 \delta^8(y) = -\kappa^2 T_2 e^{Z-2A} \delta^8(y), \quad (25)$$

$$L\Omega_7 \delta^8(y) = 2\kappa^2 T_2 \delta^8(y), \quad (26)$$

$$\begin{aligned} \left[-\frac{1}{6} \left(a - \frac{8}{3} \right) M - \frac{1}{2} \left(a + \frac{4}{3} \right) Le^Z \right] \Omega_7 \delta^8(y) \\ = -\kappa^2 T_2 \left(a + \frac{4}{3} \right) e^Z \delta^8(y), \end{aligned} \quad (27)$$

so the constants must take the values

$$L = \frac{2\kappa^2 T_2}{\Omega_7}, \quad (28)$$

$$M = 0. \quad (29)$$

The remaining integration constants are the values of the fields at infinity, X_0 , Y_0 and Z_0 , which can be rewritten in terms of A_0 , B_0 and ϕ_0 . The first two of these can be removed by constant rescaling of the coordinates, so we are left with only ϕ_0 , which is the vacuum expectation value of the dilaton field. This is exactly the standard elementary string solution [6, 10], written in arbitrary metric. We have a preserved Noether charge

$$e = \sqrt{2}\kappa T_2. \quad (30)$$

The factor 6 reflects our different normalization of H with respect to e.g. Duff et al. [10]. The mass per unit string length is

$$\mathcal{M}_2 = 2 \left(a + \frac{5}{6} \right) T_2 e^{(a+\frac{4}{3})\phi_0}, \quad (31)$$

and we can see explicitly that \mathcal{M}_2 scales with g_{MN} as it should [11].

For the n -winding state T_2 is just replaced by nT_2 everywhere, which gives exactly the mass per unit length of the $(0, n)$ winding states of Dabholkar et al. [9].

The conditions that the solution preserve half the supersymmetry can be obtained just like in the papers quoted above. We find

$$X' = 0, \quad (32)$$

$$Y' = 0, \quad (33)$$

$$\partial e^Z = \partial e^{2A+C}. \quad (34)$$

These equations are all satisfied here as we know they should be.

Provided $a + \frac{4}{3} > 0$ the solution we have found can also be interpreted as a solitonic string solution of the dual version of ten-dimensional supergravity. If we require e^{-Z} sufficiently well-behaved this solution is again unique.

4 The type I string solution

So far, we excluded $L = 0$. In this case the lhs of equation 11 has no singularity at $y = 0$ and hence we cannot have a source at the rhs. We then have to redo the analysis putting $\partial e^{2A+C} = 0$ in (8), (10), (12) and (13), and removing the rhs of (11).

We first solve equations 8-12 for $y > 0$. Equation 9 gives the same solution as before for e^X , and the remaining equations have the solution (for K different from zero)

$$Y = Y_0 + \frac{Me^{-X_0/2}}{12\sqrt{-K}} \log \left| \frac{y^6 - \sqrt{-K}e^{-X_0/2}}{y^6 + \sqrt{-K}e^{-X_0/2}} \right|, \quad (35)$$

$$Z = Z_0 + \frac{Ne^{-X_0/2}}{12\sqrt{-K}} \log \left| \frac{y^6 - \sqrt{-K}e^{-X_0/2}}{y^6 + \sqrt{-K}e^{-X_0/2}} \right|, \quad (36)$$

$$K = -\frac{M^2 + 3N^2}{7 \cdot 144} e^{-X_0}. \quad (37)$$

At $y = 0$ we now have

$$\left[-\frac{1}{6}M - \frac{1}{2}N \right] \Omega_7 \delta^8(y) = -\kappa^2 T_2 e^{Z-2A+(b-a)\phi} \delta^8(y), \quad (38)$$

$$\begin{aligned} & \left[-\frac{1}{6} \left(a - \frac{8}{3} \right) M - \frac{1}{2} \left(a + \frac{4}{3} \right) N \right] \Omega_7 \delta^8(y) \\ & = -\kappa^2 T_2 \left(b + \frac{4}{3} \right) e^{Z+(b-a)\phi} \delta^8(y). \end{aligned} \quad (39)$$

These equations have the solutions

$$M = -\frac{3}{2} \frac{\kappa^2 T_2}{\Omega_7} (b-a) e^{(Z+(b-a)\phi)(0)}, \quad (40)$$

$$N = \frac{1}{2} \frac{\kappa^2 T_2}{\Omega_7} (b-a+4) e^{(Z+(b-a)\phi)(0)}. \quad (41)$$

This is in contradiction with (13), which requires $(b-a+4)^2 + 3(b-a)^2 = 0$. However, equation 13 is the equation of motion for the string source. It can be interpreted as a no-force condition, stating that the graviton contribution to the force between two parallel source strings is cancelled by the “axion” contribution. In the present case we have no axion contribution (to the lowest order in α' at least). Furthermore, this solution does not preserve supersymmetry, nor does it have a conserved Noether charge, so there is no reason to expect it to be stable. It is hence just the configuration around a source term corresponding to an infinitely long string put in by hand. There is no reason to expect that such an unstable test source should satisfy dynamical equations of motion, or that there should be no force between two parallel unstable strings, so we just drop equation 13.

The reason we are still interested in this solution is that we are looking for just such a thing as the missing type I “soliton”, [11]. The mass per unit string length for our solution is

$$\mathcal{M}_2 = \frac{T_2}{2} \left[3 \left(a + \frac{1}{6} \right) (b - a) + \left(a + \frac{5}{6} \right) (b - a + 4) \right] e^{(b + \frac{4}{3})\phi_0} \quad (42)$$

If we choose $b = a - 2$, corresponding to a type I string source we obtain

$$\mathcal{M}_2 = -2 \left(a - \frac{1}{2} \right) e^{(a - \frac{2}{3})\phi_0}. \quad (43)$$

This is indeed the correct scaling behaviour for a type I soliton, since $a = \frac{2}{3}$ in the type I metric. The fact that there is no B_{MN} term in the source is also consistent with the type I interpretation, since B_{MN} is here a Ramond-Ramond state corresponding to a sigma model term $\gamma^{MNP} H_{MNP}$ sandwiched between two spin fields [12], and such a term should vanish in our ansatz.

For this “incomplete type I solution”, comparable to heterotic solutions where we explicitly choose not to satisfy 13, there is no alternative soliton interpretation, nor does the analysis of the zero-modes produce anything useful.

5 Final remarks

We have found the general solution to the equations of motion for ten-dimensional supergravity coupled to a three-form field strength given the standard string ansatz, and we show exactly when this solution is restricted to the fundamental (heterotic) string solution. Assuming that there is no conserved charge, we also saw that there is a possible interpretation as a type I string solution.

The calculation is easy to generalize to arbitrary $p - 1$ -brane solutions in an arbitrary dimension, see Lü et al. [8], and Tollstén [7]. The first authors limit themselves to the solution for $K < 0$, and nonzero L , for which case, however, they give the explicit solution, which is interpreted as a non-extremal black $p - 1$ -brane. Whether there exist physical interpretations for other cases remains unclear.

Acknowledgments

I am grateful for discussions with Ansar Fayyazuddin.

References

- [1] E. Witten, *Nucl. Phys. B* **443**, 85 (1995).
- [2] A.A. Tseytlin, *Phys. Lett. B* **367**, 84 (1996).

- [3] J. Polchinsky and E. Witten, *Nucl. Phys. B* **460**, 525 (1996).
- [4] A. Dabholkar, *Phys. Lett. B* **357**, 307 (1995).
- [5] C.M. Hull, *Phys. Lett. B* **357**, 545 (1995).
- [6] A. Dabholkar, G. Gibbons, J.A. Harvey and F.R. Ruiz, *Nucl. Phys. B* **340**, 33 (1990).
- [7] A.K. Tollstén, NBI-HE 96-28, hep-th/9606058.
- [8] H. Lü, C.N. Pope and K.W. Xu, *Mod. Phys. Lett. A* **11**, 1785 (1996).
- [9] A. Dabholkar, J.P. Gauntlett, J.A. Harvey and D. Waldram, *Nucl. Phys. B* **474**, 1996 (85).
- [10] M.J. Duff, Ramzi R. Khuri and J.X. Lu, *Phys. Rept* **259**, 213 (1995).
- [11] C.M. Hull, *Nucl. Phys. B* **468**, 113 (1996).
- [12] C.G. Callan, C. Lovelace, C.R. Nappi and S.A. Yost, *Nucl. Phys. B* **306**, 221 (1988).